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Some Results in Hausdorff Neutrosophic Metric Spaces on Hutchinson-Barnsley Operator

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Abstract

The main purpose of this paper is to prove the Neutrosophic contraction properties of the Hutchinson-Barnsley operator on the Neutrosophic hyperspace with respect to the Hausdorff Neutrosophic metrics. Also we discuss about the relationships between the Hausdorff Neutrosophic metrics on the Neutrosophic hyperspaces. Our theorems generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces to the Neutrosophic metric spaces.

Keywords: Contraction, Hutchinson-Barnsley Operator, Metric Space, Hausdorff Neutrosophic Metric Spaces, Hyperspace.

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1. Introduction

The Fractal Analysis was introduced by Mandelbrot in 1975 [8] and popularized by various mathematicians [6], [3], [4]. Sets with non-integral Hausdorff dimension, which exceeds its topological dimension, are called Fractals by Mandelbrot [8]. Hutchinson [6] and Barnsley [3] initiated and developed the Hutchinson-Barnsley theory (HB theory) in order to define and construct the fractal as a compact invariant subset of a complete metric space generated by the Iterated Function System (IFS) of contractions. That is, Hutchinson[6] introduced an operator on hyperspace called as Hutchinson-Barnsley operator (HB operator) to define a fractal set as a unique fixed point by using the Banach Contraction Theorem in the metric spaces. Recently in [5], [15], HB operator properties were analyzed in fuzzy metric spaces. Here we introduce the concepts and properties of HB operator in the intuitionistic fuzzy metric spaces.

Atanassov [2] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. Park [10] defined the notion of intuitionistic fuzzy metric space as a generalization of fuzzy metric space. In 1998, Smarandache [12,13] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. Basset et al. [1] Explored the neutrosophic applications in different fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making. In 2019, Kirisci et al [9] defined NMS as a generalization of IFMS and brings about fixed point theorems in complete NMS. Later Jeyaraman at el., [7, 11] proved Fixed point results in non-Archimedean generalized intuitionistic fuzzy metric spaces. In 2020, Sowndrarajan Jeyaraman and Florentin Smarandache [14] proved some fixed point results for contraction theorems in neutrosophic metric spaces.

In this paper, we prove the neutrosophic contraction properties of the HB operator on the neutrosophic hyperspace with respect to the Hausdorff neutrosophic metrics. Also we discuss about the relationships between the Hausdorff neutrosophic metrics on the neutrosophic hyperspaces. Here our theorems generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces.

2.Preliminaries

Definition 2.1. [3] Let (Σ, d) be a metric space and $\mathcal{K}_0(\Sigma)$ be the collection of all non-empty compact subsets of Σ.

Define $d(\zeta, Q) := inf_{y \in Q} d(\zeta, \eta)$ and $d(P, Q) := sup_{x \in P} d(\zeta, Q)$ for all $\zeta \in \Sigma$ and $P, Q \in \mathcal{K}_0(\Sigma)$. The Hausdorff metric or Hausdorff distance (H_d) is a function $(H_d) : \mathcal{K}_0(\Sigma) \text{x} \, \mathcal{K}_0(\Sigma) \to \mathbb{R}$ defined by $H_d(P, Q) = max\{d(P, Q), d(Q, P)\}$.

Then H_d is a metric on the hyperspace of compact sets $\mathcal{K}_0(\Sigma)$ and hence $(\mathcal{K}_0(\Sigma), H_d)$ is called a Hausdorff metric space.

Theorem 2.2. [3] If (Σ, d) is a complete metric space, then $(\mathcal{K}_0(\Sigma), H_d)$ is also a complete metric space.

Definition 2.3. [3] Let (Σ, d) be a metric space and $f_n : \Sigma \to \Sigma$, $n = 1,2,, \dots N_0 (N_0 \in \mathbb{N})$ be N_0 - contraction mappings with the corresponding contractivity ratios k_n, $n = 1,2, \dots N_0$. The system $\{\Sigma; f_n, n = 1,2, \dots N_0\}$ is called an Iterated Function System (IFS) or Hyperbolic Iterated Function System with the ratio $k = max_{n=1}^{N_0} k_n$. Then the Hutchinson Barnsley Operator (HBO) of the IFS is a function $F : \mathcal{K}_0(\Sigma) \to \mathcal{K}_0(\Sigma)$ defined by $F(Q) = \bigcup_{n=1}^{N_0} f_n(Q)$, for all $Q \in \mathcal{K}_0(\Sigma)$.

Theorem 2.4. [3] Let (Σ, d) be a metric space. Let $\{\Sigma; f_n, n = 1,2, \dots N_0; N_0 \in \mathbb{N}\}$ be an IFS. Then, the HBO (F) is a contraction mapping on $(\mathcal{K}_0(\Sigma), H_d)$.

Theorem 2.5. [3] Let (Σ, d) be a complete metric space and $\{\Sigma; f_n, n = 1,2,3 \dots N_0; N_0 \in \mathbb{N}\}$ be an IFS. Then, there exists only one compact invariant set $P_\infty \in \mathcal{K}_0(\Sigma)$ of the HBO (F) or equivalently, F has a unique fixed point namely $P_\infty \in \mathcal{K}_0(\Sigma)$.

Definition 2.6. [3] The fixed point $P_\infty \in \mathcal{K}_0(\Sigma)$ of the HBO F described in the Theorem (2.5) is called the Attractor (Fractal) of the IFS. Sometimes $P_\infty \in \mathcal{K}_0(\Sigma)$ is called as Fractal generated by the IFS and so called as IFS Fractal.

Definition 2.7. A binary operation $* : [0,1] \text{ x } [0,1] \to [0,1]$ is a continuous t-norm, if * satisfies the following conditions:

(a) * is commutative and associative;
(b) * is continuous
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2.8. A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm, if $\diamond$ satisfies the following conditions:

(a) $\diamond$ is commutative and associative;
(b) $\diamond$ is continuous
(c) $a \diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2.9. A 6-tuple $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is said to be an Neutrosophic Metric Space (shortly NMS), if Σ is an arbitrary set, $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC and Ξ, Θ *and* Υ are neutrosophic on $\Sigma \times \Sigma$ satisfying the following conditions:
For all $\zeta, \eta, \delta, \omega \in \Sigma$, $\lambda, \mu \in R^+$.

(i) $0 \leq \Xi(\zeta, \eta, \lambda) \leq 1$; $0 \leq \Theta(\zeta, \eta, \lambda) \leq 1$; $0 \leq \Upsilon(\zeta, \eta, \lambda) \leq 1$;
(ii) $\Xi(\zeta, \eta, \lambda) + \Theta(\zeta, \eta, \lambda) + \Upsilon(\zeta, \eta, \lambda) \leq 3$;
(iii) $\Xi(\zeta, \eta, \lambda) = 1$ if and only if $\zeta = \eta$;
(iv) $\Xi(\zeta, \eta, \lambda) = \Xi(\eta, \zeta, \lambda)$ for $\lambda > 0$
(v) $\Xi(\zeta, \eta, \lambda) * \Xi(\eta, \delta, \mu) \leq \Xi(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
(vi) $\Xi(\zeta, \eta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous ;
(vii) $\lim_{\lambda \to \infty} \Xi(\zeta, \eta, \lambda) = 1$ for all $\lambda > 0$;
(viii) $\Theta(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;
(ix) $\Theta(\zeta, \eta, \lambda) = \Theta(\eta, \zeta, \lambda)$ for $\lambda > 0$;
(x) $\Theta(\zeta, \eta, \lambda) \diamond \Theta(\eta, \delta, \mu) \geq \Theta(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
(xi) $\Theta(\zeta, \eta, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous;
(xii) $\lim_{\lambda \to \infty} \Theta(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;
(xiii) $\Upsilon(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;
(xiv) $\Upsilon(\zeta, \eta, \lambda) = \Upsilon(\eta, \zeta, \lambda)$ for $\lambda > 0$;
(xv) $\Upsilon(\zeta, \eta, \lambda) \diamond \Upsilon(\eta, \delta, \mu) \geq \Upsilon(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
(xvi) $\Upsilon(\zeta, \eta, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous;
(xvii) $\lim_{\lambda \to \infty} \Upsilon(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;

Then, (Ξ, Θ, Υ) is called an NMS on Σ. The functions Ξ, Θ *and* Υ denote degree of closedness, neturalness and non-closedness between ζ *and* η with respect to λ respectively.

Example 2.10. Let (Σ, d) be a metric space. Let Ξ_d, Θ_d and Υ_d be the functions defined on $\Sigma^2 \times (0, \infty)$ by $\Xi_d(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$, $\Theta_d(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}$ and $\Upsilon_d(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}$, for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi_d, \Theta_d, \Upsilon_d, *, \diamond)$ is a

NMS which is called standard NMS, and $(\Xi_d, \Theta_d, \Upsilon_d)$ is called as standard NM induced by the metric d.

Definition 2.11. Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be a NMS. The open ball $B(\zeta, r, \lambda)$ with $\zeta \in \Sigma$ and radius r, $0 < r < 1$, with respect to $\lambda > 0$ is defined as
$B(\zeta, r, \lambda) = \{\eta \in \Sigma \ : \ \Xi\ (\ \zeta, \eta, \lambda) > 1 - r, \Theta\ (\ \zeta, \eta, \lambda) < r\ and\ \Upsilon\ (\ \zeta, \eta, \lambda) < r\ \}$.
Define

$$\tau_{(\Xi,\Theta,\Upsilon)} = \left\{ \begin{matrix} P \subset \Sigma : for\ each\ \zeta \in P, \exists\ \lambda > 0 \\ and\ r \in (0,1)\ such\ that\ B(\zeta, r, \lambda)\ \subset\ P \end{matrix} \right\}.$$

Then $\tau_{(\Xi,\Theta,\Upsilon)}$ is a topology on Σ induced by a NFM (Ξ, Θ, Υ).

The topologies induced by the metric and the corresponding standard NM are the same.

Proposition 2.12. The metric space (Σ, d) is complete if and only if the standard NMS $(\Sigma, \Xi_d, \Theta_d, \Upsilon_d, *, \diamond)$is complete.

Definition 2.13. A neutrosophic fuzzy B-contraction (neutrosophic fuzzy Sehgal contraction) on an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a self –mapping f on Σ for which

$$\Xi(f(\zeta), f(\eta), k\lambda) \geq \Xi(\zeta, \eta, \lambda),\ \Theta(f(\zeta), f(\eta), k\lambda) \leq \Theta(\zeta, \eta, \lambda)\ \text{and}$$

$$\Upsilon(f(\zeta), f(\eta), k\lambda) \leq \Upsilon(\zeta, \eta, \lambda)\ \text{for all}\ \zeta, \eta \in \Sigma\ \text{and}\ \lambda > 0,$$

where k is a fixed constant in (0,1).

3. Hausdorff Neutrosophic Metric Spaces

Definition 3.1. Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be a NMS and $\tau_{(\Xi,\Theta,\Upsilon)}$ be the topology induced by the NM (Ξ, Θ, Υ).We shall denote by $\mathcal{K}_0(X)$, the set of all non-empty compact subsets of $(\Sigma, \tau_{(\Xi,\Theta,\Upsilon)})$.

Define $\Xi(\zeta, Q, \lambda) := \sup_{\eta \in Q} \Xi(\zeta, \eta, \lambda)$, $\Xi(P, Q, \lambda) := \inf_{\zeta \in P} \Xi(\zeta, Q, \lambda)$,

$\Theta(\zeta, Q, \lambda) := \inf_{\eta \in Q} \Theta(\zeta, \eta, \lambda)$, $\Theta(P, Q, \lambda) := \sup_{\zeta \in P} \Theta(\zeta, Q, \lambda)$ and

$\Upsilon(\zeta, Q, \lambda) := \inf_{\eta \in Q} \Upsilon(\zeta, \eta, \lambda)$, $\Upsilon(P, Q, \lambda) := \sup_{\zeta \in P} \Theta(\zeta, Q, \lambda)$,

for all $\zeta \in \Sigma$ and $P, Q \in \mathcal{K}_0(X)$.
Then, we define the Hausdorff NM $(H_\Xi, H_\Theta, H_\Upsilon, *, \diamond)$ as
$H_\Xi(P, Q, \lambda) = min\{\Xi(P, Q, \lambda), \Xi(Q, P, \lambda)\}$,

$H_{\Theta}(P,Q,\lambda) = max\{\Theta(P,Q,\lambda),\Theta(Q,P,\lambda)\}$and $H_{\Upsilon}(P,Q,\lambda) = max\{\Upsilon(P,Q,\lambda),\Upsilon(Q,P,\lambda)\}$.
Here $(H_{\Xi},H_{\Theta},H_{\Upsilon})$ is a NM on the hyperspace of compact sets, $\mathcal{K}_0(\Sigma)$ and hence $(\mathcal{K}_0(\Sigma),H_{\Xi},H_{\Theta},H_{\Upsilon},*,\diamond)$ is called a Hausdorff NMS.

Proposition 3.2. Let (Σ,d) be a metric space. Then the Hausdorff NM $\left(H_{\Xi_d},H_{\Theta_d},H_{\Upsilon_d}\right)$ of the standard NM $(\Xi_d,\Theta_d,\Upsilon_d)$ coincides with the standard NM $\left(H_{\Xi_d},H_{\Theta_d},H_{\Upsilon_d}\right)$ of the Hausdorff metric (H_d) on $\mathcal{K}_0(\Sigma)$.

ie., $H_{\Xi_d}(P,Q,\lambda) = \Xi_{\mathrm{H}_d}(P,Q,\lambda)$, $H_{\Theta_d}(P,Q,\lambda) = \Theta_{\mathrm{H}_d}(P,Q,\lambda)$ and

$H_{\Upsilon_d}(P,Q,\lambda) = \Upsilon_{\mathrm{H}_d}(P,Q,\lambda)$, for all $P,Q \in \mathcal{K}_0(\Sigma)$ and $\lambda > 0$.

Proof: Fix $\lambda > 0$ and $P,Q \in \mathcal{K}_0(\Sigma)$.We recall that
$\sup_{\beta \in Q} \Xi_d(\alpha,\beta,\lambda) = \frac{\lambda}{\lambda+\inf_{\beta\in Q} d(\alpha,\beta)}$, $\inf_{\beta \in Q} \Theta_d(\alpha,\beta,\lambda) = \frac{1}{1+\frac{\lambda}{\inf_{\beta\in Q} d(\alpha,\beta)}}$ and

$\inf_{\beta \in Q} \Upsilon_d(\alpha,\beta,\lambda) = \frac{1}{\frac{\lambda}{\inf_{\beta\in Q} d(\alpha,\beta)}}$, for all $\alpha \in P$.It follows that

$\Xi_d(\alpha,Q,\lambda) = \frac{\lambda}{\lambda+\mathrm{d}(\alpha,Q)}$,$\Theta_d(\alpha,Q,\lambda) = \frac{1}{1+\frac{\lambda}{\mathrm{d}(\alpha,Q)}}$and $\Upsilon_d(\alpha,Q,\lambda) = \frac{1}{\frac{\lambda}{\mathrm{d}(\alpha,Q)}}$for all $\alpha \in P$.

Then $\inf_{\alpha \in P}\Xi_d(\alpha,Q,\lambda) = \frac{\lambda}{\lambda+\sup_{\alpha\in P} \mathrm{d}(\alpha,Q)}$, $\sup_{\alpha \in P}\Theta_d(\alpha,Q,\lambda) = \frac{1}{1+\sup_{\alpha\in P}\frac{\lambda}{\mathrm{d}(\alpha,Q)}}$ and

$\sup_{\alpha \in P}\Upsilon_d(\alpha,Q,\lambda) = \frac{1}{\sup_{\alpha\in P}\frac{\lambda}{\mathrm{d}(\alpha,Q)}}$.

It follows that
$\Xi_d(P,Q,\lambda) = \frac{\lambda}{\lambda+\mathrm{d}(P,Q)}$, $\Theta_d(P,Q,\lambda) = \frac{1}{1+\frac{\lambda}{\mathrm{d}(P,Q)}} = \frac{\mathrm{d}(P,Q)}{\lambda+\mathrm{d}(P,Q)}$ and
$\Upsilon_d(P,Q,\lambda) = \frac{1}{\frac{\lambda}{\mathrm{d}(P,Q)}} = \frac{\mathrm{d}(P,Q)}{\lambda}$.
Similarly, we obtain

$\Xi_d(Q,P,\lambda) = \frac{\lambda}{\lambda+\mathrm{d}(Q,P)}$, $\Theta_d(Q,P,\lambda) = \frac{\mathrm{d}(Q,P)}{\lambda+\mathrm{d}(Q,P)}$ and $\Upsilon_d(Q,P,\lambda) = \frac{\mathrm{d}(Q,P)}{\lambda}$.

Therefore, $H_{\Xi_d}(P,Q,\lambda) = \Xi_{\mathrm{H}_d}(P,Q,\lambda)$, $H_{\Theta_d}(P,Q,\lambda) = \Theta_{\mathrm{H}_d}(P,Q,\lambda)$ and

$H_{\Upsilon_d}(P,Q,\lambda) = \Upsilon_{\mathrm{H}_d}(P,Q,\lambda)$. The proof is complete.

Using the Proposition 3.2., we can easily compute distances with respect to the Hausdorff NM $(H_{\Xi_d}, H_{\Theta_d}, H_{\Upsilon_d})$ of the standard NM $(\Xi_d, \Theta_d, \Upsilon_d)$ by computing distances with respect to the Hausdorff metric (H_d) implied by the metric d. Here, we illustrate this situation with two examples.

Example 3.3. Let $(\mathbb{R}, d)$ be the Euclidean metric space and $P = [\alpha_1, \alpha_2]$ and $Q = [\beta_1, \beta_2]$ be two compact intervals of $\mathbb{R}$. Then $d(P,Q) = |\alpha_1 - \beta_1|$ and $d(Q,P) = |\alpha_2 - \beta_2|$and hence $H_d(P,Q) = max\{|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|\}$; So, by Proposition (3.2), We have

$\Xi_d(P,Q,\lambda) = \frac{\lambda}{\lambda + max\{|\alpha_1-\beta_1|,|\alpha_2-\beta_2|\}}$, $\Theta_d(P,Q,\lambda) = \frac{max\{|\alpha_1-\beta_1|,|\alpha_2-\beta_2|\}}{\lambda + max\{|\alpha_1-\beta_1|,|\alpha_2-\beta_2|\}}$ and

$\Upsilon_d(P,Q,\lambda) = \frac{max\{|\alpha_1-\beta_1|,|\alpha_2-\beta_2|\}}{\lambda}$, for all $\lambda > 0$.

Example 3.4. Let (Σ, d) be the discrete metric space such that $|\Sigma| \geq 2$. Let P and Q be two non-empty finite subsets of Σ, with $P \neq Q$. Then $d(P,Q) = 1 = d(Q,P)$ and hence $H_d(P,Q) = 1$; so by Proposition 3.2., we have
$H_{\Xi_d}(P,Q,\lambda) = \frac{\lambda}{\lambda+1}$, $H_{\Theta_d} = \frac{1}{1+\lambda}$ and $H_{\Upsilon_d} = \frac{1}{\lambda}$, for all $\lambda > 0$.

Definition 3.5. Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be an NMS and $\tau_{(\Xi,\Theta,\Upsilon)}$ be the topology induced by (Ξ, Θ, Υ). We observe that $\left(\mathcal{K}_0(\mathcal{K}_0(\Sigma)), H_{H_\Xi}, \mathcal{H}_{H_\Theta}, \mathcal{H}_{H_\Upsilon}, *, \diamond\right)$ is also an NMS, where $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$ is the hyperspace of all non-empty compact subsets of $(\mathcal{K}_0(\Sigma), H_\Xi, H_\Theta, H_\Upsilon, *, \diamond)$ and $\left(\mathcal{H}_{H_\Xi}, \mathcal{H}_{H_\Theta}, \mathcal{H}_{H_\Upsilon}\right)$ is the Hausdorff NM on $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$ implied by the Hausdorff NM $(\Xi_d, \Theta_d, \Upsilon_d)$ on $\mathcal{K}_0(\Sigma)$. That is, for all $P \in \mathcal{K}_0(\Sigma)$ and $\mathfrak{P}, \mathfrak{Q} \in \mathcal{K}_0(\mathcal{K}_0(\Sigma))$,
$H_{H_\Xi}(\mathfrak{P},\mathfrak{Q}) = min\{H_\Xi \mathfrak{P},\mathfrak{Q}, H_\Xi(\mathfrak{Q},\mathfrak{P})\}$,$H_{H_\Theta}(\mathfrak{P},\mathfrak{Q}) = max\{H_\Theta(\mathfrak{P},\mathfrak{Q}), H_\Theta(\mathfrak{Q},\mathfrak{P})\}$
and$H_{H_\Upsilon}(\mathfrak{P},\mathfrak{Q}) = max\{H_\Upsilon(\mathfrak{P},\mathfrak{Q}), H_\Upsilon(\mathfrak{Q},\mathfrak{P})\}$ where

$$H_\Xi(\mathfrak{P},\mathfrak{Q}) := \inf_{p\in\mathfrak{P}} H_\Xi(P,\mathfrak{Q}), H_\Xi(P,\mathfrak{Q}) := \sup_{Q\in\mathfrak{Q}} H_\Xi(P,Q),$$
$$H_\Theta(\mathfrak{P},\mathfrak{Q}) := \sup_{p\in\mathfrak{P}} H_\Theta(P,\mathfrak{Q}),\ H_\Theta(P,\mathfrak{Q}) := \inf_{Q\in\mathfrak{Q}} H_\Theta(P,Q) \text{ and}$$
$$H_\Upsilon(\mathfrak{P},\mathfrak{Q}) := \sup_{p\in\mathfrak{P}} H_\Upsilon(P,\mathfrak{Q}), H_\Upsilon(P,\mathfrak{Q}) := \inf_{Q\in\mathfrak{Q}} H_\Upsilon(P,Q).$$

Proposition 3.6. Let (Σ,d) be a metric space and let $(\mathcal{K}_0(\Sigma),H_\mathrm{d})$ and $(\mathcal{K}_0(\mathcal{K}_0(\Sigma)),\mathcal{H}_{H_d})$ be the corresponding Hausdorff metric spaces. Then, the Hausdorff NM $\left(\mathcal{H}_{\Xi_{H_d}},\mathcal{H}_{\Theta_{H_d}},\mathcal{H}_{\Upsilon_{H_d}}\right)$ of the standard NM$(\Xi_{H_d},\Theta_{H_d},\Upsilon_{H_d})$ coincides with the standard NM $\left(\Xi_{\mathcal{H}_{H_d}},\Theta_{\mathcal{H}_{H_d}},\Upsilon_{\mathcal{H}_{H_d}}\right)$ of the Hausdorff metric $(\mathcal{H}_{H_d})$ on $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$, ie. $\mathcal{H}_{\Xi_{H_d}}(\mathfrak{P},\mathfrak{Q},\lambda)=\Xi_{\mathcal{H}_{H_d}}(\mathfrak{P},\mathfrak{Q},\lambda)$, $\mathcal{H}_{\Theta_{H_d}}(\mathfrak{P},\mathfrak{Q},\lambda)=\Theta_{\mathcal{H}_{H_d}}(\mathfrak{P},\mathfrak{Q},\lambda)$ and $\mathcal{H}_{\Upsilon_{H_d}}(\mathfrak{P},\mathfrak{Q},\lambda)=\Upsilon_{\mathcal{H}_{H_d}}(\mathfrak{P},\mathfrak{Q},\lambda)$ for all $\mathfrak{P},\mathfrak{Q}\in \mathcal{K}_0(\mathcal{K}_0(\Sigma))$ and $\lambda>0$.

Proof: Proposition 3.2. completes the proof.

4. Neutrosophic Hutchinson-Barnsley Operator

In this section, we define the Neutrosophic Iterated Function System (NIFS) and Neutrosophic HB Operator on the NMS.

Definition 4.1. Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$ be an NMS and $f_n:\Sigma\to\Sigma, n=1,2,3\ldots N_0 (N_0\in\mathbb{N})$ be N_0 - neutrosophic B-contractions. Then the system $\{\Sigma;\ f_n,\ n=1,2,3\ldots N_0\}$ is called a NIFS of neutrosophic B-contractions in the NMS $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$.

Definition 4.2. Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$be a NMS. Let $\{\Sigma;\ f_n,\ n=1,2,3\ldots N_0\}$ be an NIFS of neutrosophic B-contractions. Then the Neutrosophic Hutchinson-Barnsley Operator (NHBO) of the NIFS is a function $F:\mathcal{K}_0(\Sigma)\to\mathcal{K}_0(\Sigma)$ defined by $F(Q)=\cup_{n=1}^{N_0} f_n(Q)$, for all $Q\in\mathcal{K}_0(\Sigma)$.

Definition 4.3. Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$be a complete NMS. Let $f_n:\Sigma\to\Sigma, n=1,2,3\ldots N_0(N_0\in\mathbb{N})$ be a NIFS of neutrosophic B-contractions and F be the NHBO of the NIFS. We say that the set $P_\infty\in\mathcal{K}_0(\Sigma)$ is Neutrosophic Attractor (Neutrosophic Fractal) of the given NIFS, if P_∞ is a unique fixed point of the NHBO F. Such $P_\infty\in\mathcal{K}_0(\Sigma)$is also called as Fractal generated by the NIFS and so called NIFS Fractal of neutrosophic B-contractions.

Properties of NHBO

Now, we prove the interesting results about the neutrosophic B-contraction properties of operators with respect to the Hausdorff neutrosophic metric on $\mathcal{K}_0(\Sigma)$.

Theorem 4.4. Let (Σ, d) be a metric space. Let $f: \Sigma \to \Sigma$ be a contraction function on (Σ, d), with a contractivity ratio k. Then
$\mathrm{H}_{\Xi_d}(f(P), f(Q), \lambda) \geq \mathrm{H}_{\Xi_d}(\mathrm{P}, \mathrm{Q}, \lambda)$, $\mathrm{H}_{\Theta_d}(f(P), f(Q), \lambda) \leq \mathrm{H}_{\Theta_d}(\mathrm{P}, \mathrm{Q}, \lambda)$and
$\mathrm{H}_{\Upsilon_d}(f(P), f(Q), \lambda) \leq \mathrm{H}_{\Upsilon_d}(\mathrm{P}, \mathrm{Q}, \lambda)$**,** for all $P, Q \in \mathcal{K}_0(\Sigma)$ and $\lambda > 0$.

Proof: Fix $\lambda > 0$ and let $P, Q \in \mathcal{K}_0(\Sigma)$. Since f is contraction on (Σ, d) with the contractivity ratio $k \in (0,1)$ and by Theorem 2.4. for the case $\Theta = 1$, we have $H_d\big(f(P), f(Q)\big) \leq kH_d(P, Q)$. Since $\lambda > 0$ and $k \in (0,1)$,
$\frac{k\lambda}{k\lambda + H_d(f(P), f(Q))} \geq \frac{k\lambda}{k\lambda + kH_d(P,Q)} = \frac{\lambda}{\lambda + H_d(P,Q)}$, $\quad \frac{H_d(f(P),f(Q))}{k\lambda + H_d(f(P),f(Q))} \leq \frac{kH_d(P,Q)}{k\lambda + kH_d(P,Q)} = \frac{H_d(P,Q)}{\lambda + H_d(P,Q)}$
and $\frac{H_d(f(P),f(Q))}{k\lambda} \leq \frac{kH_d(P,Q)}{k\lambda} = \frac{H_d(P,Q)}{\lambda}$.
By using the above inequalities and the Proposition 3.2., we have

$$\begin{aligned}
\mathrm{H}_{\Xi_d}(f(P), f(Q), \mathrm{k}\lambda) &= \Xi_{H_d}(f(P), f(Q), \mathrm{k}\lambda) \\
&= \frac{\mathrm{k}\lambda}{\mathrm{k}\lambda + H_d\big(f(P), f(Q)\big)} \geq \frac{\lambda}{\lambda + H_d(P, Q)} \\
&= \Xi_{H_d}(P, Q, \lambda) = \mathrm{H}_{\Xi_d}(P, Q, \lambda),
\end{aligned}$$

$$\begin{aligned}
\mathrm{H}_{\Theta_d}(f(P), f(Q), \mathrm{k}\lambda) &= \Theta_{H_d}(f(P), f(Q), \mathrm{k}\lambda) \\
&= \frac{kH_d\big(f(P), f(Q)\big)}{\mathrm{k}\lambda + H_d\big(f(P), f(Q)\big)} \leq \frac{H_d(P, Q)}{\lambda + H_d(P, Q)} \\
&= \Theta_{H_d}(P, Q, \lambda) = \mathrm{H}_{\Theta_d}(P, Q, \lambda) \text{ and}
\end{aligned}$$

Similarly,
$$\begin{aligned}
\mathrm{H}_{\Upsilon_d}(f(P), f(Q), \mathrm{k}\lambda) &= \Upsilon_{H_d}(f(P), f(Q), \mathrm{k}\lambda) \\
&= \frac{kH_d\big(f(P), f(Q)\big)}{\mathrm{k}\lambda + H_d\big(f(P), f(Q)\big)} \leq \frac{H_d(P, Q)}{\lambda + H_d(P, Q)} \\
&= \Upsilon_{H_d}(P, Q, \lambda) = \mathrm{H}_{\Upsilon_d}(P, Q, \lambda).
\end{aligned}$$

The above theorem 4.4. shows that f is a neutrosophic B-contraction on $\mathcal{K}_0(\Sigma)$ with respect to the Hausdorff neutrosophic metric $\left(\mathrm{H}_{\Xi_d}, \mathrm{H}_{\Theta_d}, \mathrm{H}_{\Upsilon_d}\right)$ implied by the standard metric $(\Xi_d, \Theta_d, \Upsilon_d)$, if f is contraction on a metric space (Σ, d). The following theorem is somewhat generalization of the Theorem 4.4.

Theorem 4.5. Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a NMS. Let $(\mathcal{K}_0(\Sigma), H_\Xi, H_\Theta, H_\Upsilon, *, \diamond)$ be the corresponding Hausdorff NMS. Suppose $f: \Sigma \to \Sigma$ be a neutrosophic B-Contraction on $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. Then for $k \in (0,1)$,
$H_\Xi(f(P), f(Q), k\lambda) \geq H_\Xi(P, Q.\lambda)$, $H_\Theta(f(P), f(Q), k\lambda) \leq H_\Theta(P, Q.\lambda)$ and
$H_\Upsilon(f(P), f(Q), k\lambda) \leq H_\Upsilon(P, Q.\lambda)$ for all $P, Q \in \mathcal{K}_0(\Sigma)$ and $\lambda > 0$.

Proof: Fix $\lambda > 0$. Let $P, Q \in \mathcal{K}_0(\Sigma)$. For given $k \in (0,1)$, we get
$\Xi(f(\zeta), f(\eta), k\lambda) \geq \Xi(\zeta, \eta, \lambda)$, for all $\zeta, \eta \in \Sigma$, $\Xi(f(\zeta), f(\eta), k\lambda) \geq \Xi(\zeta, \eta, \lambda)$,
for all $\zeta \in P \; and \; \eta \in Q$,
$\sup_{\eta \in Q} \Xi(f(\zeta), f(\eta), k\lambda) \geq \sup_{y \in Q} \Xi(\zeta, \eta, \lambda)$, for all $\zeta \in P$,
$\Xi(f(\zeta), f(Q), k\lambda) \geq \Xi(\zeta, Q, \lambda)$,for all $\zeta \in P$,
$\inf_{\zeta \in P} \Xi(f(\zeta), f(Q), k\lambda) \geq \inf_{\zeta \in P} \Xi(\zeta, Q, \lambda)$, $\Xi(f(P), f(Q), k\lambda) \geq \Xi(P, Q, \lambda)$.
Similarly $\Xi(f(Q), f(\zeta), k\lambda) \geq \Xi(Q, P, \lambda)$.
Hence $H_\Xi(f(P), f(Q), k\lambda) \geq H_\Xi(P, Q.\lambda)$
Now, $\Theta(f(\zeta), f(\eta), k\lambda) \leq \Theta(\zeta, \eta, \lambda)$, for all $\zeta, \eta \in \Sigma$, $\Theta(f(\zeta), f(\eta), k\lambda) \leq \Theta(\zeta, \eta, \lambda)$,
for all $\zeta \in P \; and \; \eta \in Q$,
$\inf_{\eta \in Q} \Theta(f(\zeta), f(\eta), k\lambda) \leq \inf_{y \in Q} \Theta(\zeta, \eta, \lambda)$, for all $\zeta \in P$,
$\Theta(f(\zeta), f(Q), k\lambda) \leq \Theta(\zeta, Q, \lambda)$, for all $\zeta \in P$,
$\sup_{\zeta \in P} \Theta(f(\zeta), f(Q), k\lambda) \leq \sup_{\zeta \in P} \Theta(\zeta, Q, \lambda)$.
$\Theta(f(P), f(Q), k\lambda) \leq \Theta(P, Q, \lambda)$.
Similarly $\Theta(f(Q), f(\zeta), k\lambda) \leq \Theta(Q, P, \lambda)$.
Hence $H_\Theta(f(P), f(Q), k\lambda) \leq H_\Theta(P, Q.\lambda)$ and
$\Upsilon(f(\zeta), f(\eta), k\lambda) \leq \Upsilon(\zeta, \eta, \lambda)$, for all $\zeta, \eta \in \Sigma$, $\Upsilon(f(\zeta), f(\eta), k\lambda) \leq \Upsilon(\zeta, \eta, \lambda)$,
for all $\zeta \in P \; and \; \eta \in Q$
$\inf_{\eta \in Q} \Upsilon(f(\zeta), f(\eta), k\lambda) \leq \inf_{y \in Q} \Upsilon(\zeta, \eta, \lambda)$, for all $\zeta \in P$
$\Upsilon(f(\zeta), f(Q), k\lambda) \leq \Upsilon(\zeta, Q, \lambda)$, for all $\zeta \in P$,
$\sup_{\zeta \in P} \Upsilon(f(\zeta), f(Q), k\lambda) \leq \sup_{\zeta \in P} \Upsilon(\zeta, Q, \lambda)$.
$\Upsilon(f(P), f(Q), k\lambda) \leq \Upsilon(P, Q, \lambda)$.
Similarly $\Upsilon(f(Q), f(\zeta), k\lambda) \leq \Upsilon(Q, P, \lambda)$.
Hence $H_\Upsilon(f(P), f(Q), k\lambda) \leq H_\Upsilon(P, Q.\lambda)$. This completes the proof.

The above Theorem 4.5. shows that f is a neutrosophic B-contraction on $\mathcal{K}_0(\Sigma)$ with respect to the Hausdorff neutrosophic metric $H_{\Xi}, H_{\Theta}, H_{\Upsilon}$, if f is neutrosophic B-contraction on neutrosophic metric space $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$.

Lemma 4.6. Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be a NMS. If $Q, R \subset \Sigma$ such that $Q \subset R$, then $\Xi(\zeta, Q, \lambda) \leq \Xi(\zeta, R, \lambda)$, $\Theta(\zeta, Q, \lambda) \geq \Theta(\zeta, R, \lambda)$ and $\Upsilon(\zeta, Q, \lambda) \geq \Upsilon(\zeta, R, \lambda)$ for all $\zeta \in \Sigma$ and $\lambda > 0$.

Proof: Fix $\lambda > 0$. Let $\zeta \in \Sigma$ and $Q, R \subset \Sigma$ such that $Q \subset R$. Then,

$$\Xi(\zeta, Q, \lambda) = \sup_{q \in Q} \Xi(\zeta, q, \lambda) \leq \sup_{q \in R} \Xi(\zeta, q, \lambda) = \Xi(\zeta, R, \lambda),$$

$$\Theta(\zeta, Q, \lambda) = \inf_{q \in Q} \Theta(\zeta, q, \lambda) \geq \inf_{q \in R} \Theta(\zeta, q, \lambda) = \Theta(\zeta, R, \lambda) \text{ and}$$

$$\Upsilon(\zeta, Q, \lambda) = \inf_{q \in Q} \Upsilon(\zeta, q, \lambda) \geq \inf_{q \in R} \Upsilon(\zeta, q, \lambda) = \Upsilon(\zeta, R, \lambda).$$

Lemma 4.7. Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be a NMS. If $Q, R \subset \Sigma$ such that $Q \subset R$, then $\Xi(P, Q, \lambda) \leq \Xi(P, R, \lambda)$, $\Theta(P, Q, \lambda) \geq \Theta(P, R, \lambda)$ and $\Upsilon(P, Q, \lambda) \geq \Upsilon(P, R, \lambda)$ for all $P \subset \Sigma$ and $\lambda > 0$.

Proof: Fix $\lambda > 0$. Let $P, Q, R \subset \Sigma$ such that $Q \subset R$. By the lemma 4.6., we have

$\Xi(P, Q, \lambda) = \inf_{p \in P} \Xi(p, Q, \lambda)$, $\Xi(P, Q, \lambda) \leq \Xi(p, Q, \lambda)$, for all $p \in P$

$\Xi(P, Q, \lambda) \leq \Xi(p, R, \lambda)$, for all $p \in P$, $\Xi(P, Q, \lambda) \leq \inf_{p \in P} \Xi(p, R, \lambda)$,

$\Xi(P, Q, \lambda) \leq \Xi(P, R, \lambda)$.

Similarly, by the lemma 4.6.

$\Theta(P, Q, \lambda) = \sup_{p \in P} \Theta(p, Q, \lambda)$, $\Theta(P, Q, \lambda) \geq \Theta(p, Q, \lambda)$ for all $p \in P$,

$\Theta(P, Q, \lambda) \geq \Theta(p, R, \lambda)$ for all $p \in P$, $\Theta(P, Q, \lambda) \geq \sup_{p \in P} \Theta(p, R, \lambda)$,

$\Theta(P, Q, \lambda) \geq \Theta(P, R, \lambda)$ and

$\Upsilon(P, Q, \lambda) = \sup_{p \in P} \Upsilon(p, Q, \lambda)$, $\Upsilon(P, Q, \lambda) \geq \Upsilon(p, Q, \lambda)$ for all $p \in P$,

$\Upsilon(P, Q, \lambda) \geq \Upsilon(p, R, \lambda)$ for all $p \in P$, $\Upsilon(P, Q, \lambda) \geq \sup_{p \in P} \Upsilon(p, R, \lambda)$

$\Upsilon(P, Q, \lambda) \geq \Upsilon(P, R, \lambda)$.

Lemma 4.8. Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be a NMS. If $P, Q, R \subset \Sigma$, then

$\Xi(P \cup Q, R, \lambda) = min\{\Xi(P, R, \lambda), \Xi(Q, R, \lambda)\}$, $\Theta(P \cup Q, R, \lambda) = max\{\Theta(P, R, \lambda), \Theta(Q, R, \lambda)\}$ and $\Upsilon(P \cup Q, R, \lambda) = max\{\Upsilon(P, R, \lambda), \Upsilon(Q, R, \lambda)\}$,for all $\lambda > 0$.

Proof: Fix $\lambda > 0$. Let $P, Q, R \subset \Sigma$. Then

$$\Xi(P \cup Q, R, \lambda) = \inf_{\zeta \in P \cup Q} \Xi(\zeta, R, \lambda) = min\left\{\inf_{p \in P} \Xi(p, R, \lambda), \inf_{q \in Q} \Xi(q, R, \lambda),\right\}$$
$$= min\{\Xi(P, R, \lambda), \Xi(Q, R, \lambda)\},$$
$$\Theta(P \cup Q, R, \lambda) = \sup_{\zeta \in P \cup Q} \Theta(\zeta, R, \lambda) = max\left\{\sup_{p \in P} \Theta(p, R, \lambda), \sup_{q \in Q} \Theta(q, R, \lambda),\right\}$$
$$= max\{\Theta(P, R, \lambda), \Theta(Q, R, \lambda)\} \text{ and}$$
$$\Upsilon(P \cup Q, R, \lambda) = \sup_{\zeta \in P \cup Q} \Upsilon(\zeta, R, \lambda) = max\left\{\sup_{p \in P} \Upsilon(p, R, \lambda), \sup_{q \in Q} \Upsilon(q, R, \lambda),\right\}$$
$$= max\{\Upsilon(P, R, \lambda), \Upsilon(Q, R, \lambda)\}$$

Lemma 4.9. Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a NMS. Let $(\mathcal{K}_0(\Sigma), H_\Xi, H_\Theta, H_\Upsilon, *, \diamond)$ be the corresponding Hausdorff NMS. If $P, Q, R, S \in \mathcal{K}_0(\Sigma)$ then

$H_\Xi(P \cup Q, R \cup S, \lambda) \geq min\{H_\Xi(P, R, \lambda), H_\Xi(Q, S, \lambda)\}$,
$H_\Theta(P \cup Q, R \cup S, \lambda) \leq max\{H_\Theta(P, R, \lambda), H_\Theta(Q, S, \lambda)\}$ and
$H_\Upsilon(P \cup Q, R \cup S, \lambda) \leq max\{H_\Upsilon(P, R, \lambda), H_\Upsilon(Q, S, \lambda)\}$, for all $\lambda > 0$.

Proof: Fix $\lambda > 0$. Let $P, Q, R, S \in \mathcal{K}_0(\Sigma)$.
By using Lemma 4.7. and Lemma 4.8., we get

$$\Xi(P \cup Q, R \cup S, \lambda) = min\{\Xi(P, R \cup S, \lambda), \Xi(Q, R \cup S, \lambda)\}$$
$$\geq min\{\Xi(P, R, \lambda), \Xi(Q, S, \lambda)\}$$
$$\geq min\{H_\Xi(P, R, \lambda), H_\Xi(Q, S, \lambda)\}.$$

Similarly, $\Xi(R \cup S, P \cup Q, \lambda) \geq min\{H_\Xi(P, R, \lambda), H_\Xi(Q, S, \lambda)\}$.
Hence, $H_\Xi(P \cup Q, R \cup S, \lambda) \geq min\{H_\Xi(P, R, \lambda), H_\Xi(Q, S, \lambda)\}$.

$$\Theta(P \cup Q, R \cup S, \lambda) = max\{\Theta(P, R \cup S, \lambda), \Theta(Q, R \cup S, \lambda)\}$$
$$\leq max\{\Theta(P, R, \lambda), \Theta(Q, S, \lambda)\}$$
$$\leq max\{H_\Theta(P, R, \lambda), H_\Theta(Q, S, \lambda)\}.$$

Similarly, $\Theta(R \cup S, P \cup Q, \lambda) \leq max\{H_\Theta(P, R, \lambda), H_\Theta(Q, S, \lambda)\}$.
Hence, $H_\Theta(R \cup S, P \cup Q, \lambda) \leq max\{H_\Theta(P, R, \lambda), H_\Theta(Q, S, \lambda)\}$ and

$$\Upsilon(P \cup Q, R \cup S, \lambda) = max\{\Upsilon(P, R \cup S, \lambda), \Upsilon(Q, R \cup S, \lambda)\}$$
$$\leq max\{\Upsilon(P, R, \lambda), \Upsilon(Q, S, \lambda)\}$$
$$\leq max\{H_\Upsilon(P, R, \lambda), H_\Upsilon(Q, S, \lambda)\}.$$

Similarly, $\Upsilon(R \cup S, P \cup Q, \lambda) \leq max\{H_\Upsilon(P, R, \lambda), H_\Upsilon(Q, S, \lambda)\}$.
Hence, $H_\Upsilon(R \cup S, P \cup Q, \lambda) \leq max\{H_\Upsilon(P, R, \lambda), H_\Upsilon(Q, S, \lambda)\}$. This completes the proof.

The following theorem is a generalized version of the Theorem 4.5.

Theorem 4.10. Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$ be a NMS. Let $(\mathcal{K}_0(\Sigma),H_\Xi,H_\Theta,H_\Upsilon,*,\diamond)$ be the corresponding Hausdorff NMS. Suppose $f_n:\Sigma\to\Sigma, n=1,2,\dots N_0; N_0\in\mathbb{N}$, is a neutrosophic B-Contraction on $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$. Then the neutrosophic HBO is also a neutrosophic B-Contraction on $(\mathcal{K}_0(\Sigma),H_\Xi,H_\Theta,H_\Upsilon,*,\diamond)$.

Proof: Fix $\lambda>0$. Let $P,Q\in\mathcal{K}_0(\Sigma)$. By using the Lemma 4.9. and Theorem 4.5. for a given $k\in(0,1)$, we get

$$H_\Xi(F(P),F(Q),k\lambda)=H_\Xi\left(\bigcup_{n=1}^{N_0} f_n(P),\bigcup_{n=1}^{N_0} f_n(Q),k\lambda\right)$$

$$\geq \min_{n=1}^{N_0} H_\Xi(f_n(P),f_n(Q),k\lambda)\geq H_\Xi(P,Q,\lambda),$$

$$H_\Theta(F(P),F(Q),k\lambda)=H_\Theta\left(\bigcup_{n=1}^{N_0} f_n(P),\bigcup_{n=1}^{N_0} f_n(Q),k\lambda\right)$$

$$\leq \max_{n=1}^{N_0} H_\Theta(f_n(P),f_n(Q),k\lambda)\leq H_\Theta(P,Q,\lambda) \text{ and}$$

$$H_\Upsilon(F(P),F(Q),k\lambda)=H_\Upsilon\left(\bigcup_{n=1}^{N_0} f_n(P),\bigcup_{n=1}^{N_0} f_n(Q),k\lambda\right)$$

$$\leq \max_{n=1}^{N_0} H_\Upsilon(f_n(P),f_n(Q),k\lambda)\leq H_\Upsilon(P,Q,\lambda).$$ This completes the proof.

From the above Theorem 4.10., we conclude that the operator F is a neutrosophic B-contraction on $\mathcal{K}_0(\Sigma)$ with respect to the Hausdorff neutrosophic metric $(H_\Xi,H_\Theta,H_\Upsilon)$, if f_n is neutrosophic B-contraction on an neutrosophic metric space$(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$ for each $n\in\{1,2,\dots N_0\}$.

5. Hausdorff Neutrosophic Metrics On $\mathcal{K}_0(\Sigma)$ and $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$

Now, we investigate the relationships between the hyperspaces $\mathcal{K}_0(\Sigma)$ and $\mathcal{K}_0(\mathcal{K}_0(\Sigma))$ and the Hausdorff neutrosophic metrics H_Ξ and $\mathcal{H}_{H_\Xi}$.

Theorem 5.1. Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond)$ be a NMS. Let $(\mathcal{K}_0(\Sigma),H_\Xi,H_\Theta,H_\Upsilon,*,\diamond)$ and $\left(\mathcal{K}_0(\mathcal{K}_0(\Sigma)),\mathcal{H}_{H_\Xi},\mathcal{H}_{H_\Theta},\mathcal{H}_{H_\Upsilon},*,\diamond\right)$ be the corresponding Hausdorff Neutrosophic hyper spaces. Let $\mathfrak{P},\mathfrak{Q}\in\mathcal{K}_0(\mathcal{K}_0(\Sigma))$ be such that $\{p\in P: P\in\mathfrak{P}\},\{q\in Q, Q\in\mathfrak{Q}\}\in\mathcal{K}_0(\Sigma)$. Then $H_\Xi(\{p\in P: P\in\mathfrak{P}\},\{q\in Q: Q\in\mathfrak{Q}\},\lambda)\geq\mathcal{H}_{H_\Xi}(\mathfrak{P},\mathfrak{Q},\lambda)$,

$H_\Theta(\{p\in P: P\in\mathfrak{P}\},\{q\in: Q\in\mathfrak{Q}\},\lambda)\leq\mathcal{H}_{H_\Theta}(\mathfrak{P},\mathfrak{Q},\lambda)$ and

$H_\Upsilon(\{p\in P: P\in\mathfrak{P}\},\{q\in Q: Q\in\mathfrak{Q}\},\lambda)\leq\mathcal{H}_{H_\Upsilon}(\mathfrak{P},\mathfrak{Q},\lambda)$ for all $\lambda>0$.

Proof: Fix $\lambda>0$. Firstly, we note that

$$\begin{aligned}\Xi(Q,\{p\in P: P\in\mathfrak{P}\},\lambda) &= \inf_{q\in Q}\Xi(q,\{p\in P: P\in\mathfrak{P}\},\lambda)\\ &= \inf_{q\in Q}\sup_{\{p\in P: P\in\mathfrak{P}\}}\Xi(q,p,\lambda)\\ &= \inf_{q\in Q}\sup_{P\in\mathfrak{P}}\sup_{p\in P}\Xi(q,p,\lambda)\geq\sup_{P\in\mathfrak{P}}\inf_{q\in Q}\sup_{p\in P}\Xi(q,p,\lambda)\\ &= \sup_{P\in\mathfrak{P}}\Xi(Q,P,\lambda).\end{aligned}$$

It follows that

$$\begin{aligned}\Xi(\{q\in Q, Q\in\mathfrak{Q}\},\{p\in P: P\in\mathfrak{P}\},\lambda) &= \inf_{\{q\in Q: Q\in\mathfrak{Q}\}}\Xi(q,\{p\in P: P\in\mathfrak{P}\},\lambda)\\ &= \inf_{Q\in\mathfrak{Q}}\inf_{q\in Q}\Xi(q,\{p\in P: P\in\mathfrak{P}\},\lambda) = \inf_{Q\in\mathfrak{Q}}\Xi(Q,\{p\in P: P\in\mathfrak{P}\},\lambda)\\ &\geq \inf_{Q\in\mathfrak{Q}}\sup_{P\in\mathfrak{P}}\Xi(Q,P,\lambda).\end{aligned}$$

Similarly, $\Xi(\{p\in P: P\in\mathfrak{P}\},\{q\in Q, Q\in\mathfrak{Q}\},\lambda)\geq\inf_{P\in\mathfrak{P}}\sup_{Q\in\mathfrak{Q}}\Xi(P,Q,\lambda)$.

Hence, $H_\Xi(\{p\in P: P\in\mathfrak{P}\},\{q\in Q, Q\in\mathfrak{Q}\},\lambda)$

$$\begin{aligned}&= min\left\{\begin{matrix}\Xi(\{p\in P: P\in\mathfrak{P}\},\{q\in Q, Q\in\mathfrak{Q}\},\lambda),\\ \Xi(\{\{q\in Q, Q\in\mathfrak{Q}\},\{p\in P: P\in\mathfrak{P}\},\lambda)\end{matrix}\right\}\\ &\geq min\left\{\inf_{P\in\mathfrak{P}}\sup_{Q\in\mathfrak{Q}}\Xi(P,Q,\lambda),\inf_{Q\in\mathfrak{Q}}\sup_{P\in\mathfrak{P}}\Xi(Q,P,\lambda)\right\}\\ &\geq min\left\{\inf_{P\in\mathfrak{P}}\sup_{Q\in\mathfrak{Q}}H_\Xi(P,Q,\lambda),\inf_{Q\in\mathfrak{Q}}\sup_{P\in\mathfrak{P}}H_\Xi(Q,P,\lambda)\right\}\\ &= min\{H_\Xi(\mathfrak{P},\mathfrak{Q},\lambda),H_\Xi(\mathfrak{Q},\mathfrak{P},\lambda)\}\\ &= \mathcal{H}_{H_\Xi}(\mathfrak{P},\mathfrak{Q},\lambda).\end{aligned}$$

Secondly, we note that

$$\Theta(Q,\{p\in P: P\in\mathfrak{P}\},\lambda) = \sup_{q\in Q}\Theta(q,\{p\in P: P\in\mathfrak{P}\},\lambda)$$

$$= \sup_{q \in Q} \inf_{\{p \in P: P \in \mathfrak{P}\}} \Theta(q,p,\lambda) = \sup_{q \in Q} \inf_{P \in \mathfrak{P}} \inf_{p \in P} \Theta(q,p,\lambda)$$
$$\leq \inf_{P \in \mathfrak{P}} \sup_{q \in Q} \inf_{p \in P} \Theta(q,p,\lambda) = \inf_{P \in \mathfrak{P}} \Theta(Q,P,\lambda).$$

It follows that

$$\Theta(\{q \in : Q \in \mathfrak{Q}\}, \{p \in P: P \in \mathfrak{P}\}, \lambda) = \sup_{\{q \in Q: Q \in \mathfrak{Q}\}} \Theta(q, \{p \in P: P \in \mathfrak{P}\}, \lambda)$$
$$= \sup_{Q \in \mathfrak{Q}} \sup_{q \in Q} \Theta(q, \{p \in P: P \in \mathfrak{P}\}, \lambda) = \sup_{Q \in \mathfrak{Q}} \Theta(Q, \{p \in P: P \in \mathfrak{P}\}, \lambda)$$
$$\leq \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} \Theta(Q,P,\lambda).$$

Similarly, $\Theta(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q: Q \in \mathfrak{Q}\}, \lambda) \leq \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} \Theta(P,Q,\lambda)$.

Hence, $H_\Theta(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q, Q \in \mathfrak{Q}\}, \lambda)$

$$= max \begin{Bmatrix} \Theta(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q: Q \in \mathfrak{Q}\}, \lambda), \\ \Theta(\{\{q \in Q: Q \in \mathfrak{Q}\}, \{p \in P: P \in \mathfrak{P}\}, \lambda)\end{Bmatrix}$$
$$\leq max \left\{ \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} \Theta(P,Q,\lambda), \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} \Theta(Q,P,\lambda) \right\}$$
$$\leq max \left\{ \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} H_\Theta(P,Q,\lambda), \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} H_\Theta(Q,P,\lambda) \right\}$$
$$= max\{H_\Theta(\mathfrak{P}, \mathfrak{Q}, \lambda), H_\Theta(\mathfrak{Q}, \mathfrak{P}, \lambda)\}$$
$$= \mathcal{H}_{H_\Theta}(\mathfrak{P}, \mathfrak{Q}, \lambda) \text{ and}$$

Lastly, we note that

$$\Upsilon(Q, \{p \in P: P \in \mathfrak{P}\}, \lambda) = \sup_{q \in Q} \Upsilon(q, \{p \in P: P \in \mathfrak{P}\}, \lambda)$$
$$= \sup_{q \in Q} \inf_{\{p \in P: P \in \mathfrak{P}\}} \Upsilon(q,p,\lambda) = \sup_{q \in Q} \inf_{P \in \mathfrak{P}} \inf_{p \in P} \Upsilon(q,p,\lambda)$$
$$\leq \inf_{P \in \mathfrak{P}} \sup_{q \in Q} \inf_{p \in P} \Upsilon(q,p,\lambda) = \inf_{P \in \mathfrak{P}} \Upsilon(Q,P,\lambda)$$

It follows that

$$\Upsilon(\{q \in : Q \in \mathfrak{Q}\}, \{p \in P: P \in \mathfrak{P}\}, \lambda) = \sup_{\{q \in Q: Q \in \mathfrak{Q}\}} \Upsilon(q, \{p \in P: P \in \mathfrak{P}\}, \lambda)$$
$$= \sup_{Q \in \mathfrak{Q}} \sup_{q \in Q} \Upsilon(q, \{p \in P: P \in \mathfrak{P}\}, \lambda) = \sup_{Q \in \mathfrak{Q}} \Upsilon(Q, \{p \in P: P \in \mathfrak{P}\}, \lambda)$$
$$\leq \sup_{Q \in \mathfrak{Q}} \inf_{P \in \mathfrak{P}} \Upsilon(Q,P,\lambda).$$

Similarly, $\Upsilon(\{p \in P: P \in \mathfrak{P}\}, \{q \in : Q \in \mathfrak{Q}\}, \lambda) \leq \sup_{P \in \mathfrak{P}} \inf_{Q \in \mathfrak{Q}} \Upsilon(P,Q,\lambda)$.

Hence, $H_\Upsilon(\{p \in P: P \in \mathfrak{P}\}, \{q \in Q, Q \in \mathfrak{Q}\}, \lambda)$

$$= max\begin{Bmatrix} \Upsilon(\{p \in P : P \in \mathfrak{P}\}, \{q \in Q : Q \in \mathfrak{Q}\}, \lambda), \\ \Upsilon(\{\{q \in Q : Q \in \mathfrak{Q}\}, \{p \in P : P \in \mathfrak{P}\}, \lambda) \end{Bmatrix}$$

$$\leq max\left\{ \underset{P \in \mathfrak{P}}{\sup} \underset{Q \in \mathfrak{Q}}{inf} \Upsilon(P, Q, \lambda), \underset{Q \in \mathfrak{Q}}{\sup} \underset{P \in \mathfrak{P}}{inf} \Upsilon(Q, P, \lambda) \right\}$$

$$\leq max\left\{ \underset{P \in \mathfrak{P}}{\sup} \underset{Q \in \mathfrak{Q}}{inf} H_\Upsilon(P, Q, \lambda), \underset{Q \in \mathfrak{Q}}{\sup} \underset{P \in \mathfrak{P}}{inf} H_\Upsilon(Q, P, \lambda) \right\}$$

$$= max\{H_\Upsilon(\mathfrak{P}, \mathfrak{Q}, \lambda), H_\Upsilon(\mathfrak{Q}, \mathfrak{P}, \lambda)\} = \mathcal{H}_{H_\Upsilon}(\mathfrak{P}, \mathfrak{Q}, \lambda).$$

The proof is complete.

6. Conclusions

In this paper, we proved the neutrosophic contraction properties of the Hutchinson-Barnsley operator on the neutrosophic hyperspace with respect to the Hausdorff neutrosophic metrics. Also we discussed about the relationships between the Hausdorff neutrosophic metrics on the neutrosophic hyperspaces. This paper will lead our direction to develop the Hutchinson-Barnsley Theory in the sense of neutrosophic B-contractions in order to define a fractal set in the neutrosophic metric spaces as a unique fixed point of the Neutrosophic HBO.

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